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Zagreb Indices of Graphs with Added Edges

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Abstract

Edge deletion and addition to a graph is an important combinatorial method in Graph Theory which enables one to calculate some properties of a graph by means of similar graphs. In this paper, as a sequel to a recent paper on edge deletion, we consider the change in the first and second Zagreb indices of a simple graph G when an arbitrary edge is added. This can be used to calculate the first and second Zagreb indices of larger graphs in terms of the Zagreb indices of smaller graphs. As some examples, some inequalities for the change of Zagreb indices for path, cycle, star, complete, complete bipartite and tadpole graphs are given.

1 Introduction

1 2 3

Let $G = (V, E)$ be a simple graph with $\mid V(G) \mid= n$ vertices and $\mid E(G) \mid= m$ edges. That is, we do not allow loops or multiple edges. For a vertex $v \in V(G)$, we denote the degree of v by $d_G(v)$. A vertex with degree one is called a pendant vertex. Similarly, we shall use the term "pendant edge" for an edge having a pendant vertex. As usual, we denote by $P_n, C_n, S_n, K_n, K_{r,s}$ and $T_{r,s}$ the path, cycle, star, complete, complete bipartite and tadpole graphs, respectively.

Topological graph indices are defined and used in many areas to study several properties of different objects such as atoms and molecules. Several topological graph indices have been defined

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and studied by many mathematicians and chemists as most graphs are generated from molecules by replacing atoms with vertices and bonds with edges. Two of the most important topological graph indices are called the first and second Zagreb indices denoted by $M_1(G)$ and $M_2(G)$, respectively:

$$M_1(G) = \sum_{u \in V(G)} d_G^2(u) \quad \text{and} \quad M_2(G) = \sum_{uv \in E(G)} d_G(u) d_G(v). \tag{1}$$

They were first defined in 1972 by Gutman and Trinajstic, [3], and are referred to due to their uses in QSAR and QSPR studies. In [1], some results on the first Zagreb index together with some other indices are given. In [2], the multiplicative versions of these indices are studied. For graph operations, these indices are calculated in [4]. Some relations between Zagreb indices and some other indices such as ABC, GA and Randic indices are obtained in [5]. Zagreb indices of subdivision graphs were studied in [7] and these were calculated for the line graphs of the subdivision graphs in [6]. A more generalized version of subdivision graphs is called r-subdivision graphs and Zagreb indices of r-subdivision graphs are calculated in [8]. These indices are calculated for several important graph classes in [9].

In [10], the same authors obtained some formulae and inequalities for the change of the first and second Zagreb indices when one or more edges are deleted from a given graph.

In this paper, we calculate the change of the first and second Zagreb indices when a new edge e is added to the given graph. We specially require that both the graphs G and $G + \{e\}$ to be simple. One can use this results to obtain the Zagreb indices of graphs with $n + 1$ vertices by means of the Zagreb indices of graphs with n vertices where these n vertices are selected from the set of $n + 1$ vertices, or vice versa.

Finally we give examples to the change of Zagreb indices for some well-known graphs when a new edge is added.

2 Adding a new edge to a graph

In this section, we will determine the amount of change in the first and second Zagreb indices when a new edge is added to any simple graph not necessarily connected:

Theorem 2.1. *Let G be a simple graph. Let us add an edge e to form a larger graph $G + \{e\}$.*

i) *If the added edge e is a pendant edge connecting the vertex v_i of degree d_i in G with a new pendant vertex v_{n+1} of degree $d_{n+1} = 1$, then*

$$M_1(G + \{e\}) = M_1(G) + 2(d_i + 1). \tag{2}$$

ii) *If the added edge is not a pendant edge, in other words, any two vertices v_i, v_j with degrees d_i, d_j of the graph G, respectively, are connected by a new edge e, then*

$$M_1(G + \{e\}) = M_1(G) + 2(d_i + d_j + 1). \tag{3}$$

Proof. **i)** Let the graph G have vertices $v_1, v_2, \cdots, v_n$, with degrees $d_1, d_2, \cdots, d_n$, respectively. Let us connect the vertex $v_i \in G$ with a new vertex v_{n+1} of degree 1 which is not a vertex of G, by means of an edge e. Recall that $M_1(G) = \sum_{k=1}^{n} d_k^2$.

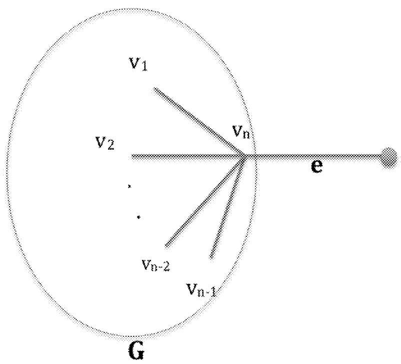

Figure 1: Adding a pendant edge e to the graph G

Since we add a pendant edge, $d_{n+1} = 1$. So

$$\begin{aligned} M_1(G + \{e\}) &= \sum_{\substack{k=1 \\ k \neq i}}^{n} d_k^2 + (d_i + 1)^2 + 1^2 \\ &= \sum_{k=1}^{n} d_k^2 + 2d_i + 2. \end{aligned}$$

ii) Let the added edge e be a non-pendant edge and let it connect two vertices v_i, v_j of G.

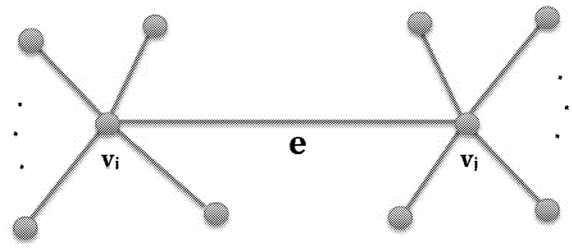

Figure 2: Adding a non-pendant edge e to the graph G

In that case, only the degrees d_i and d_j of v_i and v_j increase by 1 each, respectively. All other degrees remain the same. Therefore

$$\begin{aligned} M_1(G+\{e\}) &= \sum_{\substack{k=1 \\ k\neq i,j}}^{n} d_k^2 + (d_i+1)^2 + (d_j+1)^2 \\ &= \sum_{k=1}^{n} d_k^2 + 2(d_i+d_j+1) \\ &= M_1(G) + 2(d_i+d_j+1). \end{aligned}$$

Hence the result follows. □

Secondly, we consider the change in the second Zagreb index:

Theorem 2.2. *Let G be a simple graph. Let us add the edge e to form a larger graph $G+\{e\}$.*

i) *If the added edge e is a pendant edge connecting the vertex v_i of degree d_i in G with a new vertex v_{n+1} of degree $d_{n+1}=1$, then*

$$M_2(G+\{e\}) = M_2(G) + d_1 + d_2 + \cdots + d_i + 1. \tag{4}$$

ii) *Let G have m edges. If the added edge e is not a pendant edge, in other words, if e connects two vertices of the graph G, let us say v_1 and v_2 with degrees d_1 and d_2, respectively, then*

$$M_2(G+\{e\}) = M_2(G) + 2m + d_1d_2 + 1. \tag{5}$$

Proof. **i)** Let us represent the neighbouring vertices of v_i in G by $v_1, v_2, \cdots, v_{i-1}$. As the added edge e is a pendant edge connecting the vertex v_i of degree d_i in G with a new vertex v_{n+1} of degree $d_{n+1} = 1$, d_i will increase by 1. So

$$\begin{aligned} M_2(G+\{e\}) &= \sum_{\substack{v_kv_j\in E(G)\\ k,j\neq i}}^{n} d_kd_j + (d_i+1)d_1 + (d_i+1)d_2 + \cdots + (d_i+1)d_{i-1} + 1\cdot(d_i+1) \\ &= \sum_{v_kv_j\in E(G)}^{n} d_kd_j + 1\cdot d_i + 1(d_1+d_2+\cdots+d_{i-1}) + 1 \\ &= M_2(G) + d_1 + d_2 + \cdots + d_i + 1. \end{aligned}$$

ii) Let the added edge e be a non-pendant edge and let it connect two vertices v_1, v_2 of G.

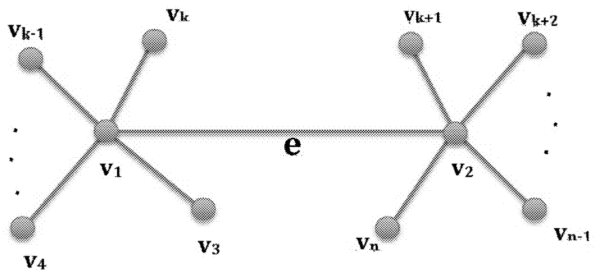

Figure 3: Adding a new edge e combining two vertices of the graph G

Let us represent the neighbouring vertices of v_1 by $v_3, v_4, \cdots, v_k$, and the neighbouring vertices of v_2 by $v_{k+1}, v_{k+2}, \cdots, v_n$. Some of these vertices may coincide. So

$$\begin{aligned} M_2(G+\{e\}) &= M_2(G) + (d_1+1)(d_2+1) + 1(d_3+d_4+\cdots+d_k) + 1(d_{k+1}+\cdots+d_n) \\ &= M_2(G) + (d_1+1)(d_2+1) + d_3 + d_4 + \cdots + d_k + d_{k+1} + \cdots + d_n \\ &= M_2(G) + \sum_{i=1}^{n} d_i + d_1\cdot d_2 + 1 \\ &= M_2(G) + 2m + d_1\cdot d_2 + 1. \end{aligned}$$

Hence the result follows. □

3 Adding an edge to some well-known graphs

In this section we consider six well-known graph types, namely path P_n, cycle C_n, star S_n, complete K_n, complete bipartite $K_(r,s)$, tadpole $T_(r,s)$, and calculate the change in their first and second Zagreb indices by adding an arbitrary edge. There are different types of edges with different vertex degrees. So the choice of the edge to be added is important and effects the change in the Zagreb indices.

Theorem 3.1. *If an edge e is added to the path graph P_n, then*

$$4 \leq M_1(P_n + \{e\}) - M_1(P_n) \leq 6$$

and

$$4 \leq M_2(P_n + \{e\}) - M_2(P_n) \leq 7.$$

That is, adding any edge to a path graph increases the first Zagreb index by a number between 4 and 6, and the second Zagreb index by a number between 4 and 7, respectively.

Proof. There are two types of edges in P_n: two end edges with vertex degrees 1 and 2, and $n-3$ middle edges with both vertex degrees 2. If we add a pendant edge, say e, to one of the two end vertices, then

$$M_1(P_n + \{e\}) = 2 \cdot 1^2 + (n-2+1) \cdot 2^2 = 4n - 2$$

and the change in the first Zagreb index is

$$M_1(P_n + \{e\}) - M_1(P_n) = (4n-2) - (4n-6) = 4.$$

If we add a new pendant edge, say e to one of the middle vertices, then

$$M_1(P_n + \{e\}) = 3 \cdot 1^2 + (n-3) \cdot 2^2 + 1 \cdot 3^2 = 4n$$

and the change in the first Zagreb index is

$$M_1(P_n + \{e\}) - M_1(P_n) = 4n - (4n-6) = 6.$$

Secondly, let us calculate the second Zagreb index for $P_n + \{e\}$. If a new edge e is added to one of the two end vertices, then

$$M_2(P_n + \{e\}) = 2 \cdot (1 \cdot 2) + (n-2) \cdot (2 \cdot 2) = 4n - 4$$

and therefore $M_2(P_n + \{e\}) - M_2(P_n) = 4$. If a new edge e is added to one of the middle vertices, then

$$M_2(P_n + \{e\}) = 2 \cdot (1 \cdot 2) + 1 \cdot (1 \cdot 3) + (n-5) \cdot (2 \cdot 2) = 4n - 1$$

which implies that $M_2(P_n - \{e\}) - M_2(P_n) = 7$. Hence the proof is complete. □

Theorem 3.2. *If an edge e is added to cycle graph C_n, then*

$$6 \leq M_1(C_n + \{e\}) - M_1(C_n) \leq 10,$$

and

$$7 \leq M_2(C_n + \{e\}) - M_2(C_n) \leq 17.$$

Proof. If we add a pendant edge, say e, we get

$$M_1(C_n + \{e\}) = (n-1) \cdot 2^2 + 1 \cdot 3^2 + 1 \cdot 1^2 = 4n + 6$$

which implies that $M_1(C_n + \{e_1\}) - M_1(C_n) = 6$. If we add an edge which is not pendant, say e, we get

$$M_1(C_n + \{e\}) = (n-2) \cdot 2^2 + 2 \cdot 3^2 = 4n + 10.$$

Secondly, let us calculate the second Zagreb index for $C_n + \{e\}$. If e is a pendant edge, then

$$M_2(C_n + \{e\}) = (n-2) \cdot (2 \cdot 2) + 2 \cdot (3 \cdot 2) + 1 \cdot (3 \cdot 1) = 4n + 7$$

and therefore $M_2(C_n + \{e\}) - M_2(C_n) = 7$. If e is the added edge which is not pendant, then

$$M_2(C_n + \{e\}) = (n-4) \cdot (2 \cdot 2) + 4 \cdot (3 \cdot 2) + 1 \cdot (3 \cdot 3) = 4n + 17$$

which implies that $M_2(C_n + \{e\}) - M_2(C_n) = 17$. Hence we complete the proof. □

The following results can similarly be shown.

Theorem 3.3. *If any edge e is added to S_n, then*

$$n^2 - n + 4 \leq M_1(S_n + \{e\}) - M_1(S_n) \leq n^2 + n,$$

and

$$n^2 - n + 2 \leq M_2(S_n + \{e\}) - M_2(S_n) \leq n^2.$$

Theorem 3.4. *If any edge e is added to K_n, then*

$$M_1(K_n + \{e\}) - M_1(K_n) = 2n,$$

and

$$M_2(K_n + \{e\}) - M_2(K_n) = n^2 - n + 1.$$

Theorem 3.5. *If any edge e is added to $K_{r,s}$, then*

$$4(r+s) + 6 \leq M_1(T_{r,s} + \{e\}) - M_1(T_{r,s}) \leq 4(r+s) + 14$$

and

$$4(r+s) + 8 \leq M_2(T_{r,s} + \{e\}) - M_2(T_{r,s}) \leq 4(r+s) + 30.$$

Theorem 3.6. *If any edge e is added to $T_{r,s}$, then*

$$M_1(K_{r,s} + \{e\}) - M_1(K_{r,s}) = 2s + 2$$

and

$$M_2(K_{r,s} + \{e\}) - M_2(K_{r,s}) = rs + s + 1.$$

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